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FRACTAL SERIES AND INFINITE PRODUCTS

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RESUMEN.

Se muestra en este artículo un algoritmo para asociar conjuntos fractales a ciertas series de vectores en $\mathbb{R}^n$ y a ciertos productos infinitos reales o complejos. Se muestra la conexión existente entre algunos casos particulares de estos fractales y ciertos fractales obtenidos por M. F. Barnsley desde un punto de vista dinámico, como atractores de sistemas de funciones iterados. Se obtienen las dimensiones de Hausdorff y la función de dimensión de algunos fractales generados por series, y se construyen ejemplos de interés en teoría de la medida. El contenido de este artículo fué presentado como ponencia en la European Conference on Iterations que se celebró en Batschuns (Austria) en Septiembre de 1989, y posteriormente ha sido aceptada para su publicación en las memorias del citado congreso.

1. INTRODUCTION.

We show in this paper a natural algorithm to associate fractal sets to certain series of vectors in $\mathbb{R}^n$ and to certain real or complex infinite products. We show the connexion between some particular cases of these fractal sets and certain fractal sets studied by M.F.Barnsley from a dynamic point of view as attractors of "Iteration Function Systems". We obtain the Hausforff dimension and the dimension function of some fractal sets obtained with fractal series which lead to interesting examples from the point of view of measure theory. Mathematical background for this paper can be found in refs. [4],[5] and [6].

2. DEFINITIONS AND NOTATIONS.

Let $\sum_{i=0}^{\infty} a_i$ be an absolutely convergent series of vectors in $\mathbb{R}^n$, where n is the least natural number such that $\mathbb{R}^n$ can contain all terms of the series, except perhaps for a finite number of them, that we can assume removed from the series if necessary. We call associated set to Σa_i the set C of all possible sums of terms of the series:

$$C = \{ \sum_{i\in I} a_i : I \subseteq \mathbb{N} \}.$$

We assume that the sum over the empty set of indices is zero. It can be proved that C is a compact subset of $\mathbb{R}^n$.

We call $\mathfrak{P}_j$ the set of all subsets of $\{0,1,2\ldots,j\}$ and C_j the set of all possible sums of terms of the series which do not contain any term a_i with $i \leq j$. Then

$$C = \bigcup_{I\in\mathfrak{P}_j} \{ \sum_{i\in I} a_i + C_{j+1} \},$$

and we call j-standard covering of C the right hand member of the above equation. This equation shows that C is made up of copies of C_j translated by vectors $\sum_{i\in I} a_i$, $I\in\mathfrak{P}_j$, some of which may coincide. We call ρ_j the cardinal of this set of vectors. Then ρ_j coincides with the number of different copies of C_j which made up C.

We define the series $\Sigma\, a_i$ as a fractal series when C has continuum power and Lebesgue n-dimensional measure 0, that is, when we may expect C to behave as a fractal.

Finally we call R_j the sum $\sum_{i<j} |a_i|$.

3. TWO THEOREMS ABOUT FRACTAL SERIES.

The following theorem gives sufficient conditions for fractal series.

THEOREM 1.

$\Sigma\, a_i$ is a fractal series if $a_i \neq 0$ for infinitely many indices, and some of the following conditions are satisfied:

i) $\liminf \rho_i |C_{i+1}|^n = 0$,

ii) $\liminf 2^i |C_{i+1}|^n = 0$,

iii) $\liminf 2^i R_i^n = 0$.

Taking into account that $|2R_i| \geq |C_{i+1}|$, it is clear that i) $\Rightarrow$ ii) $\Rightarrow$ iii).

These criteria can be used to prove the next theorem, which shows that many fractal series exist:

THEOREM 2

Let f be an analytic and non polynomial real function in an open set $D \subseteq \mathbb{R}$, and let $\Sigma\, a_i(x-x_0)^i$ be the expansion of f in a power series about $x_0 \in D$, with radius of convergence ρ. Then for all $t \neq x_0$ and $t \in B(x_0, \rho/2)$, $\Sigma\, a_i(t-x_0)^i$ is a fractal series. If f is is complex and $\rho \leq 1$, then $\Sigma a_i(t-x_0)^i$ is fractal for all $t \in B(x_0, \sqrt{\rho/2}\,) - \Delta$, where Δ is either x_0 or a finite set of straight lines through x_0.

The proofs of these theorems can be found in ref.[1].
In figs. 1 to 6 we can see some fractal sets associated to binomial series, obtained expanding in power series about 0 the complex functions $f(x) = (1+x)^\alpha$, $\alpha \in \mathbb{R}$, $\alpha < 0$.

4. A DYNAMIC APPROACH.

The complex function $f(x) = (1-x)^{-1}$ has Σx^i as expansion in a power series about 0. The associated set to this series for different values of x (fig. 2 is a picture of this kind of set) has been studied by M. Barnsley from a dynamic point of view as attractors of an "Iterated Function System".

An I.F.S. is a complete metric space X together with a finite set $\{w_1, w_2 \dots, w_n\}$ of contractions $w_i : X \longrightarrow X.$, $1 \le i \le n$.

In the set $\mathcal{K}(X)$ of all non empty compact subsets of X we define the Hausdoff metric: for all $A, B \in \mathcal{K}(X)$,

$$d_H(A,B) = \max \left(\max_{x \in B}\{d(x,A)\}, \max_{x \in A}\{d(x,B)\}\right).$$

$\mathcal{K}(X)$ is a complete metric space with this metric, and the mapping $W: \mathcal{K}(X) \longrightarrow \mathcal{K}(X)$

$$W(A) = \bigcup_{i=1}^{n} w_i(A)$$

is a contraction of $\mathcal{K}(X)$, so that there exists a compact subset $\mathbb{A}$ of X which is the only fixed point of W. In the Hausdorff metric $\lim W^n(E) = \mathbb{A}$ for all $E \in \mathcal{K}(X)$. The set $\mathbb{A}$ is the attractor of the I.F.S.

In our case, let X be the complex plane $\mathbb{C}$ and w_1, w_2 contractions with equations $w_1(x) = tx$, $w_2(x) = 1 + tx$, with $t \in \mathbb{C} - \{0\}$ and $|t| < 1$. Then, the attractor of this I.F.S. is the set C associated to the series Σt^i. Hence

$$C = w_1(C) \cup w_2(C),$$

so that, if $|t|$ is sufficiently small (in order to avoid the overlapping of the two sets in the right hand side of the above equation), C is a selfsimilar set. A more extensive study of this topic can be found in refs. [2], [3].

5. HAUSDORFF DIMENSION OF ASSOCIATED SETS TO FRACTAL SERIES.

For series other than $\Sigma\ t^i$, the associated sets are not selfsimilar. Theorem 3 gives an upper bound for the Hausdorff dimension of any associated set to a series, and a lower bound which coincides with the upper one under certain assumption for the series. The proof of this theorem involves two previous lemmas, and it can be seen in ref. [1].

THEOREM 3

Let $\Sigma\ a_i$ be an a. c. series of vectors in $\mathbb{R}^n$, C its associated set and Dim(C) the Hausdorff dimension of C. Then

i) $\mathrm{Dim}(C) \leq \liminf -\log \rho_i/R_i = s$

ii) If $\limsup R_i/|a_i| < 1$, $\mathrm{Dim}(C) \leq s$.

With this theorem can be proved that the Hausdorff dimension of the associated sets to the series obtained by expanding in power series the functions exp(x), sin(x), sinh(x), cos(x), cosh(x) is 0 for any real or complex value of x .

The dimension of the associated set to the complex binomial series $\Sigma \binom{\alpha}{i} x^i$ is $\log 2/\log(1/|x|)$ for all $x \neq 0$ and $|x|$ sufficiently small.

Let C_n be the associated set to the series $\Sigma\ t^{2^n i}/(2^n i)!$. For all n $C_n \supset C_{n+1}$. C_0 is the associated set to the exponential series. Thus all these sets have Hausdorff dimension 0. Moreover, each C_n is a continuum power disjoint union of translated copies of C_{n+1}, so that, for any Borel measure λ invariant under translations and such that $0 < \lambda(C_n) < \infty$, $\lambda(C_{n+1}) = 0$ and $\lambda(C_{n-1}) = \infty$.

For any real number $s < 1$ there exist chains of fractal sets with the same properties as above and with Hausdorff dimension s. The examples can be obtained by using fractal series (see ref.[1]). It seems likely that this result remains valid for all $s \geq 1$.

These examples show that the system of usual Hausdorff measures, which has a measure H^s for each real number s, is not, in general, the adequate system to measure associated sets to series, since the H^s measure is a Borel measure invariant under translations.

The system of generalized Hausdorff measures (see ref. [4]) has a measure H^f for each "dimension function " f which belongs to a certain set $\mathcal{H}$ of dimension functions such that $f \in \mathcal{H} \leftrightarrow f$ is a real function defined for all $s \geq 0$, monotonic increasing, positive for $t > 0$, and continuous to the right. The subset of functions of $\mathcal{H}$ such that $f(0) = 0$ is called $\mathcal{H}_0$. The following theorem gives criteria to find the suitable dimension function for the associated set to a series.

THEOREM 4

Let $f \in \mathcal{H}_0$. Then

i) $\liminf f(2R_i)\rho_i = s \Rightarrow H^f(C) \leq s$

ii) $\limsup |a_i|/R_i < 1$ and $\liminf f(R_i)\rho_i > 0 \rightarrow H^f(C) > 0$

COROLLARY

Let $f \in \mathcal{H}_0$, $\limsup |a_i|/R_i < 1$ and $0 < \liminf f(R_i)2^i = s < \infty$. Then $0 < H^f(C) \leq 2s$.

So, if the hypothese of the collorary are satisfied, f is a suitable dimension function for C.

This theorem can be used to get suitable dimension functions for associated sets to the binomial series. Let $C_{\alpha,t}$ be the associated set to the series $\Sigma \binom{\alpha}{i} t^i$. We know by Theorem 3 that, if $|t|$ is small enough and $\alpha \leq -1$,

$$\mathrm{Dim}(C_{\alpha,t}) = \log 2/\log(1/|t|).$$

We see that this dimension does not depend on α. Writing s_t for $\mathrm{Dim}(C_{\alpha,t})$, one can check that the dimension function

$$f_{\alpha,t}(x) = x^{s_t}.(-\log x)^{(\alpha+1)s_t}$$

satisfies the hypothese of the above corollary. Therefore $0 < f_{\alpha,t}(C_{\alpha,t}) < \infty$. Moreover, $\alpha' < \alpha \Rightarrow$

$$\lim_{x\to 0} f_{\alpha',t}(x)/f_{\alpha,t}(x) = 0.$$

From this it can be deduced (see ref [4]) that $H^{f_{\alpha,t}}(C_{\alpha',t}) = \infty$ and $H^{f_{\alpha',t}}(C_{\alpha,t}) = 0$. Thus, the family of fractal sets $\{C_{\alpha,t}\}_{\alpha\le -1}$ is a family of sets with continuum power, all of them with the same Hausdorff dimension s_t, but with different suitable dimension functions which can be totally ordered, and which also allow us the total ordering of the corresponding sets in the obvious way.

These results pose several natural problems. Is it possible to construct chains like those above with continumm-many sets?. Is there any system of Borel measures invariant under translations that give finite and positive measure to "most" sets with a given Hausdorff dimension ? (" most" could mean that it is not possible to construct a chain like those above within any family of sets which has measure either 0 or ∞ for any measure of the system).

The parameter α in the system of measures $H^{f_{\alpha,t}}$ allows us to divide a given Hausdorff dimension s (at least if s < 1) in continuum many "subdimensions". Is it possible to construct chains like those above with sets of a given subdimension ?.

6. FRACTAL INFINITE PRODUCTS.

Let $\Pi\,(1 + a_i)$ be a convergent real or complex infinite product. In a obvious way its associated set C, the j-standard covering, etc, can be defined. An infinite product is said to be fractal if its associated set C has continuum power and the Lebesgue n dimensional measure of C is 0, where n = 1 if the infinite product is real and n = 2 if it is complex.

THEOREM 5

i) If $a_i \neq 0$ for infinitely many indices, C has continuum power.

ii) $Dim(C) = Dim(C')$ where C' is the associated set to the series $\Sigma \log(1+a_i)$.

iii) If the hypothesis of i) holds and $\Sigma\ a_i$ satisfies the hypothesis iii) of Theorem 1, then $\Pi\ (1+a_i)$ is a fractal infinite product.

We can see in figs. 7 to 10 some fractal sets associated to infinite products with $a_i = z^i$, for different complex values of z. In figs. 11 to 18 associated sets to infinite products $\Pi(u+z^i)$ are represented for different complex values of u , all of them with $|u| = 1$, and for different complex or real values of z. This kind of infinite product is not convergent, but their partial products have a finite set of accumulation points if u has rational argument, and a whole circle of accumulation points if u has irrational argument.

M.MORAN, MADRID, SEPTEMBER 1989

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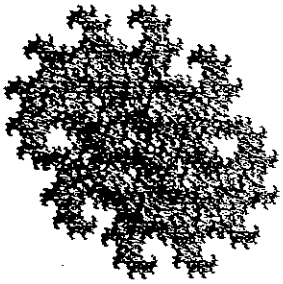

Fig. 1
$f(z)=(1+z)^{-2}$, $z=.66_{90}$

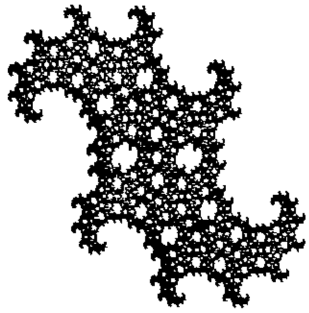

Fig. 2
$f(z)=(1+z)^{-1}$, $z=.7_{90}$

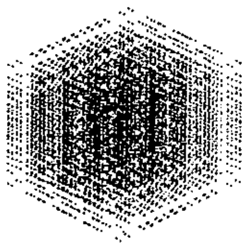

Fig. 3
$f(z)=(1+z)^{-5}$, $z=.7_{60}$

Fig. 4
$f(z)=(1+z)^{-5}$, $z=.66_{90}$

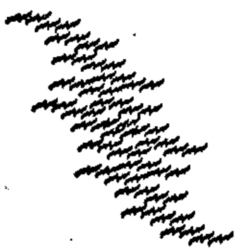

Fig. 5
$f(z)=(1+z)^{-2}$, $z=.55_{90}$

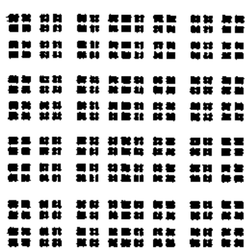

Fig. 6
$f(z)=(1+z)^{-2}$, $z=.55_{90}$

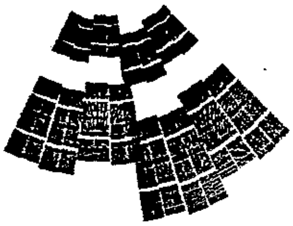

Fig. 7

$\Pi(1+z^n)$, $z=.7_{90}$

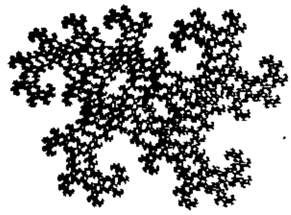

Fig. 8

$\Pi(1+z^n)$, $z=.7_{40}$

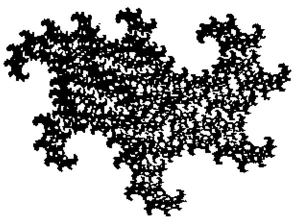

Fig. 9

$\Pi(1+z^n)$, $z=.7_{30}$

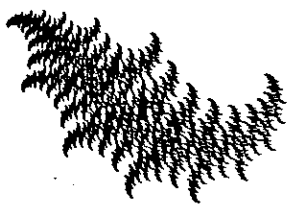

Fig. 10

$\Pi(1+z^n)$, $z=.7_{15}$

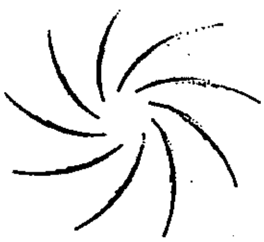

Fig. 11

$\Pi(1_{40}+.7_0{}^n)$

Fig. 12

$\Pi(1_{72}+.7_0{}^n)$

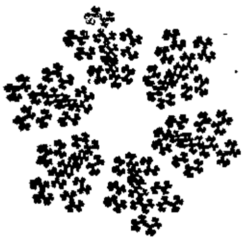

Fig. 13

$\Pi(1_{60}+.66_{60}^{n})$

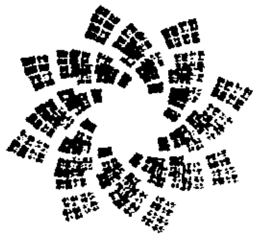

Fig. 14

$\Pi(1_{45}+.66_{90}^{n})$

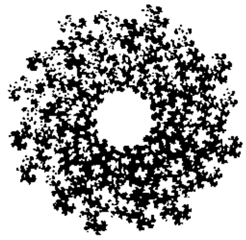

Fig. 15

$\Pi(1_{45}+.7_{45}^{n})$

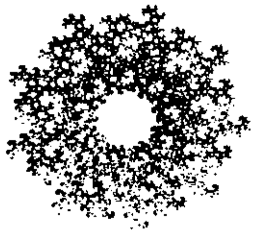

Fig. 16

$\Pi(1_{40}+.7_{40}^{n})$

Fig. 17

$\Pi(1_{60}+.7_{15}^{n})$

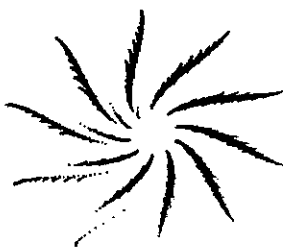

Fig. 18

$\Pi(1_{36}+.7_{5}^{n})$